

# Relativistic treatment in $D$ -dimensions to a spin-zero particle with noncentral equal scalar and vector ring-shaped Kratzer potential

Sameer M. Ikhdaire\* and Ramazan Sever†

*\*Department of Physics, Near East University, Nicosia, North Cyprus, Mersin 10, Turkey*

*†Department of Physics, Middle East Technical University, 06531 Ankara, Turkey.*

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## Abstract

The Klein-Gordon equation in  $D$ -dimensions for a recently proposed Kratzer potential plus ring-shaped potential is solved analytically by means of the conventional Nikiforov-Uvarov method. The exact energy bound-states and the corresponding wave functions of the Klein-Gordon are obtained in the presence of the noncentral equal scalar and vector potentials. The results obtained in this work are more general and can be reduced to the standard forms in three-dimensions given by other works.

Keywords: Energy eigenvalues and eigenfunctions, Klein-Gordon equation, Kratzer potential, ring-shaped potential, non-central potentials, Nikiforov and Uvarov method.

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\*sikhdaire@neu.edu.tr

†sever@metu.edu.tr

## I. INTRODUCTION

In various physical applications including those in nuclear physics and high energy physics [1,2], one of the interesting problems is to obtain exact solutions of the relativistic equations like Klein-Gordon and Dirac equations for mixed vector and scalar potential. The Klein-Gordon and Dirac wave equations are frequently used to describe the particle dynamics in relativistic quantum mechanics. The Klein-Gordon equation has also been used to understand the motion of a spin-0 particle in large class of potentials. In recent years, much efforts have been paid to solve these relativistic wave equations for various potentials by using different methods. These relativistic equations contain two objects: the four-vector linear momentum operator and the scalar rest mass. They allow us to introduce two types of potential coupling, which are the four-vector potential (V) and the space-time scalar potential (S).

Recently, many authors have worked on solving these equations with physical potentials including Morse potential [3], Hulthen potential [4], Woods-Saxon potential [5], Pösch-Teller potential [6], reflectionless-type potential [7], pseudoharmonic oscillator [8], ring-shaped harmonic oscillator [9],  $V_0 \tanh^2(r/r_0)$  potential [10], five-parameter exponential potential [11], Rosen-Morse potential [12], and generalized symmetrical double-well potential [13], etc. It is remarkable that in most works in this area, the scalar and vector potentials are almost taken to be equal (i.e.,  $S = V$ ) [2,14]. However, in some few other cases, it is considered the case where the scalar potential is greater than the vector potential (in order to guarantee the existence of Klein-Gordon bound states) (i.e.,  $S > V$ ) [15-19]. Nonetheless, such physical potentials are very few. The bound-state solutions for the last case is obtained for the exponential potential for the  $s$ -wave Klein-Gordon equation when the scalar potential is greater than the vector potential [15].

The study of exact solutions of the nonrelativistic equation for a class of non-central potentials with a vector potential and a non-central scalar potential is of considerable interest in quantum chemistry [20-22]. In recent years, numerous studies [23] have been made in

analyzing the bound states of an electron in a Coulomb field with simultaneous presence of Aharonov-Bohm (AB) [24] field, and/or a magnetic Dirac monopole [25], and Aharonov-Bohm plus oscillator (ABO) systems. In most of these works, the eigenvalues and eigenfunctions are obtained by means of separation of variables in spherical or other orthogonal curvilinear coordinate systems. The path integral for particles moving in non-central potentials is evaluated to derive the energy spectrum of this system analytically [26]. In addition, the idea of SUSY and shape invariance is also used to obtain exact solutions of such non-central but separable potentials [27,28]. Very recently, the conventional Nikiforov-Uvarov (NU) method [29] has been used to give a clear recipe of how to obtain an explicit exact bound-states solutions for the energy eigenvalues and their corresponding wave functions in terms of orthogonal polynomials for a class of non-central potentials [30].

Another type of noncentral potentials is the ring-shaped Kratzer potential, which is a combination of a Coulomb potential plus an inverse square potential plus a noncentral angular part [31,32]. The Kratzer potential has been used to describe the vibrational-rotational motion of isolated diatomic molecules [33] and has a mixed-energy spectrum containing both bound and scattering states with bound-states have been widely used in molecular spectroscopy [34]. The ring-shaped Kratzer potential consists of radial and angular-dependent potentials and is useful in studying ring-shaped molecules [22]. In taking the relativistic effects into account for spin-0 particle in the presence of a class of noncentral potentials, Yasuk *et al* [35] applied the NU method to solve the Klein-Gordon equation for the noncentral Coulombic ring-shaped potential [21] for the case  $V = S$ . Further, Berkdemir [36] also used the same method to solve the Klein-Gordon equation for the Kratzer-type potential.

Recently, Chen and Dong [37] proposed a new ring-shaped potential and obtained the exact solution of the Schrödinger equation for the Coulomb potential plus this new ring-shaped potential which has possible applications to ring-shaped organic molecules like cyclic polyenes and benzene. This type of potential used by Chen and Dong [37] appears to be very similar to the potential used by Yasuk *et al* [35]. Moreover, Cheng and Dai [38], proposed a new potential consisting from the modified Kratzer's potential [33] plus the

new proposed ring-shaped potential in [37]. They have presented the energy eigenvalues for this proposed exactly-solvable non-central potential in three dimensional (i.e.,  $D = 3$ )-Schrödinger equation by means of the NU method. The two quantum systems solved by Refs [37,38] are closely relevant to each other as they deal with a Coulombic field interaction except for a slight change in the angular momentum barrier acts as a repulsive core which is for any arbitrary angular momentum  $\ell$  prevents collapse of the system in any dimensional space due to the slight perturbation to the original angular momentum barrier. Very recently, we have also applied the NU method to solve the Schrödinger equation in any arbitrary  $D$ -dimension to this new modified Kratzer-type potential [39,40].

The aim of the present paper is to consider the relativistic effects for the spin-0 particle in our recent works [39,40]. So we want to present a systematic recipe to solving the  $D$ -dimensional Klein-Gordon equation for the Kratzer plus the new ring-shaped potential proposed in [38] using the simple NU method. This method is based on solving the Klein-Gordon equation by reducing it to a generalized hypergeometric equation.

This work is organized as follows: in section II, we shall present the Klein-Gordon equation in spherical coordinates for spin-0 particle in the presence of equal scalar and vector noncentral Kratzer plus the new ring-shaped potential and we also separate it into radial and angular parts. Section III is devoted to a brief description of the NU method. In section IV, we present the exact solutions to the radial and angular parts of the Klein-Gordon equation in  $D$ -dimensions. Finally, the relevant conclusions are given in section V.

## II. THE KLEIN-GORDON EQUATION WITH EQUAL SCALAR AND VECTOR POTENTIALS

In relativistic quantum mechanics, we usually use the Klein-Gordon equation for describing a scalar particle, i.e., the spin-0 particle dynamics. The discussion of the relativistic behavior of spin-zero particles requires understanding the single particle spectrum and the

exact solutions to the Klein Gordon equation which are constructed by using the four-vector potential  $\mathbf{A}_\lambda$  ( $\lambda = 0, 1, 2, 3$ ) and the scalar potential ( $S$ ). In order to simplify the solution of the Klein-Gordon equation, the four-vector potential can be written as  $\mathbf{A}_\lambda = (A_0, 0, 0, 0)$ . The first component of the four-vector potential is represented by a vector potential ( $V$ ), i.e.,  $A_0 = V$ . In this case, the motion of a relativistic spin-0 particle in a potential is described by the Klein-Gordon equation with the potentials  $V$  and  $S$  [1]. For the case  $S \geq V$ , there exist bound-state (real) solutions for a relativistic spin-zero particle [15-19]. On the other hand, for  $S = V$ , the Klein-Gordon equation reduces to a Schrödinger-like equation and thereby the bound-state solutions are easily obtained by using the well-known methods developed in nonrelativistic quantum mechanics [2].

The Klein-Gordon equation describing a scalar particle (spin-0 particle) with scalar  $S(r, \theta, \varphi)$  and vector  $V(r, \theta, \varphi)$  potentials is given by [2,14]

$$\left\{ \mathbf{P}^2 - [E_R - V(r, \theta, \varphi)/2]^2 + [\mu + S(r, \theta, \varphi)/2]^2 \right\} \psi(r, \theta, \varphi) = 0, \quad (1)$$

where  $E_R$ ,  $\mathbf{P}$  and  $\mu$  are the relativistic energy, momentum operator and rest mass of the particle, respectively. The potential terms are scaled in (1) by Alhaidari *et al* [14] so that in the nonrelativistic limit the interaction potential becomes  $V$ .

In this work, we consider the equal scalar and vector potentials case, that is,  $S(r, \theta, \varphi) = V(r, \theta, \varphi)$  with the recently proposed general non-central potential taken in the form of the Kratzer plus ring-shaped potential [38-40]:

$$V(r, \theta, \varphi) = V_1(r) + \frac{V_2(\theta)}{r^2} + \frac{V_3(\varphi)}{r^2 \sin^2 \theta}, \quad (2)$$

$$V_1(r) = -\frac{A}{r} + \frac{B}{r^2}, \quad V_2(\theta) = C \tan^2 \theta, \quad V_3(\varphi) = 0, \quad (3)$$

where  $A = 2a_0 r_0$ ,  $B = a_0 r_0^2$  and  $C$  is positive real constant with  $a_0$  is the dissociation energy and  $r_0$  is the equilibrium internuclear distance [33]. The potentials in Eq. (3) introduced by Cheng-Dai [38] reduce to the Kratzer potential in the limiting case of  $C = 0$  [33]. In fact the energy spectrum for this potential can be obtained directly by considering it as special case of the general non-central separable potentials [30].

In the relativistic atomic units ( $\hbar = c = 1$ ), the  $D$ -dimensional Klein-Gordon equation in (1) becomes [41]

$$\left\{ \frac{1}{r^{D-1}} \frac{\partial}{\partial r} \left( r^{D-1} \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right] - (E_R + \mu) \left( V_1(r) + \frac{V_2(\theta)}{r^2} + \frac{V_3(\varphi)}{r^2 \sin^2 \theta} \right) + (E_R^2 - \mu^2) \right\} \psi(r, \theta, \varphi) = 0. \quad (4)$$

with  $\psi(r, \theta, \varphi)$  being the spherical total wave function separated as follows

$$\psi_{njm}(r, \theta, \varphi) = R(r) Y_j^m(\theta, \varphi), \quad R(r) = r^{-(D-1)/2} g(r), \quad Y_j^m(\theta, \varphi) = H(\theta) \Phi(\varphi). \quad (5)$$

Inserting Eqs (3) and (5) into Eq. (4) and using the method of separation of variables, the following differential equations are obtained:

$$\frac{1}{r^{D-1}} \frac{d}{dr} \left( r^{D-1} \frac{dR(r)}{dr} \right) - \left[ \frac{j(j+D-2)}{r^2} + \alpha_2^2 \left( \alpha_1^2 - \frac{A}{r} + \frac{B}{r^2} \right) \right] R(r) = 0, \quad (6)$$

$$\left[ \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d}{d\theta} \right) - \frac{m^2 + C\alpha_2^2 \cos^2 \theta}{\sin^2 \theta} + j(j+D-2) \right] H(\theta) = 0, \quad (7)$$

$$\frac{d^2 \Phi(\varphi)}{d\varphi^2} + m^2 \Phi(\varphi) = 0, \quad (8)$$

where  $\alpha_1^2 = \mu - E_R$ ,  $\alpha_2^2 = \mu + E_R$ ,  $m$  and  $j$  are constants and with  $m^2$  and  $\lambda_j = j(j+D-2)$  are the separation constants.

For a nonrelativistic treatment with the same potential, the Schrödinger equation in spherical coordinates is

$$\left\{ \frac{1}{r^{D-1}} \frac{\partial}{\partial r} \left( r^{D-1} \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right] + 2\mu \left[ E_{NR} - V_1(r) - \frac{V_2(\theta)}{r^2} - \frac{V_3(\varphi)}{r^2 \sin^2 \theta} \right] \right\} \psi(r, \theta, \varphi) = 0. \quad (9)$$

where  $\mu$  and  $E_{NR}$  are the reduced mass and the nonrelativistic energy, respectively. Besides, the spherical total wave function appearing in Eq. (9) has the same representation as in Eq.

(5) but with the transformation  $j \rightarrow \ell$ . Inserting Eq. (5) into Eq. (9) leads to the following differential equations [39,40]:

$$\frac{1}{r^{D-1}} \frac{d}{dr} \left( r^{D-1} \frac{dR(r)}{dr} \right) - \left[ \frac{\lambda_D}{r^2} - 2\mu \left( E_{NR} + \frac{A}{r} - \frac{B}{r^2} \right) \right] R(r) = 0, \quad (10)$$

$$\left[ \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d}{d\theta} \right) - \frac{m^2 + 2\mu C \cos^2 \theta}{\sin^2 \theta} + \ell(\ell + D - 2) \right] H(\theta) = 0, \quad (11)$$

$$\frac{d^2 \Phi(\varphi)}{d\varphi^2} + m^2 \Phi(\varphi) = 0, \quad (12)$$

where  $m^2$  and  $\lambda_\ell = \ell(\ell + D - 2)$  are the separation constants. Equations (6)-(8) have the same functional form as Eqs (10)-(12). Therefore, the solution of the Klein-Gordon equation can be reduced to the solution of the Schrödinger equation with the appropriate choice of parameters:  $j \rightarrow \ell$ ,  $\alpha_1^2 \rightarrow -E_{NR}$  and  $\alpha_2^2 \rightarrow 2\mu$ .

The solution of Eq. (8) is well-known periodic and must satisfy the period boundary condition  $\Phi(\varphi + 2\pi) = \Phi(\varphi)$  which is the azimuthal angle solution:

$$\Phi_m(\varphi) = \frac{1}{\sqrt{2\pi}} \exp(\pm im\varphi), \quad m = 0, 1, 2, \dots \quad (13)$$

Additionally, Eqs (6) and (7) are radial and polar angle equations and they will be solved by using the Nikiforov-Uvarov (NU) method [29] which is given briefly in the following section.

### III. NIKIFOROV-UVAROV METHOD

The NU method is based on reducing the second-order differential equation to a generalized equation of hypergeometric type [29]. In this sense, the Schrödinger equation, after employing an appropriate coordinate transformation  $s = s(r)$ , transforms to the following form:

$$\psi_n''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)} \psi_n'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)} \psi_n(s) = 0, \quad (14)$$

where  $\sigma(s)$  and  $\tilde{\sigma}(s)$  are polynomials, at most of second-degree, and  $\tilde{\tau}(s)$  is a first-degree polynomial. Using a wave function,  $\psi_n(s)$ , of the simple ansatz:

$$\psi_n(s) = \phi_n(s)y_n(s), \quad (15)$$

reduces (14) into an equation of a hypergeometric type

$$\sigma(s)y_n''(s) + \tau(s)y_n'(s) + \lambda y_n(s) = 0, \quad (16)$$

where

$$\sigma(s) = \pi(s) \frac{\phi(s)}{\phi'(s)}, \quad (17)$$

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s), \quad \tau'(s) < 0, \quad (18)$$

and  $\lambda$  is a parameter defined as

$$\lambda = \lambda_n = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s), \quad n = 0, 1, 2, \dots \quad (19)$$

The polynomial  $\tau(s)$  with the parameter  $s$  and prime factors show the differentials at first degree be negative. It is worthwhile to note that  $\lambda$  or  $\lambda_n$  are obtained from a particular solution of the form  $y(s) = y_n(s)$  which is a polynomial of degree  $n$ . Further, the other part  $y_n(s)$  of the wave function (14) is the hypergeometric-type function whose polynomial solutions are given by Rodrigues relation

$$y_n(s) = \frac{B_n}{\rho(s)} \frac{d^n}{ds^n} [\sigma^n(s)\rho(s)], \quad (20)$$

where  $B_n$  is the normalization constant and the weight function  $\rho(s)$  must satisfy the condition [29]

$$\frac{d}{ds}w(s) = \frac{\tau(s)}{\sigma(s)}w(s), \quad w(s) = \sigma(s)\rho(s). \quad (21)$$

The function  $\pi$  and the parameter  $\lambda$  are defined as

$$\pi(s) = \frac{\sigma'(s) - \tilde{\tau}(s)}{2} \pm \sqrt{\left(\frac{\sigma'(s) - \tilde{\tau}(s)}{2}\right)^2 - \tilde{\sigma}(s) + k\sigma(s)}, \quad (22)$$



$$\lambda = k + \pi'(s). \quad (23)$$

In principle, since  $\pi(s)$  has to be a polynomial of degree at most one, the expression under the square root sign in (22) can be arranged to be the square of a polynomial of first degree [29]. This is possible only if its discriminant is zero. In this case, an equation for  $k$  is obtained. After solving this equation, the obtained values of  $k$  are substituted in (22). In addition, by comparing equations (19) and (23), we obtain the energy eigenvalues.

#### IV. EXACT SOLUTIONS OF THE RADIAL AND ANGLE-DEPENDENT EQUATIONS

##### A. Separating variables of the Klein-Gordon equation

We seek to solving the radial and angular parts of the Klein-Gordon equation given by Eqs (6) and (7). Equation (6) involving the radial part can be written simply in the following form [39-41]:

$$\frac{d^2 g(r)}{dr^2} - \left[ \frac{(M-1)(M-3)}{4r^2} - \alpha_2^2 \left( \frac{A}{r} - \frac{B}{r^2} \right) + \alpha_1^2 \alpha_2^2 \right] g(r) = 0, \quad (24)$$

where

$$M = D + 2j. \quad (25)$$

On the other hand, Eq. (7) involving the angular part of Klein-Gordon equation retakes the simple form

$$\frac{d^2 H(\theta)}{d\theta^2} + ctg(\theta) \frac{dH(\theta)}{d\theta} - \left[ \frac{m^2 + C\alpha_2^2 \cos^2 \theta}{\sin^2 \theta} - j(j + D - 2) \right] H(\theta) = 0. \quad (26)$$

Thus, Eqs (24) and (26) have to be solved latter through the NU method in the following subsections.

## B. Eigenvalues and eigenfunctions of the angle-dependent equation

In order to apply NU method [29,30,33,35,36,38-40,42-44], we use a suitable transformation variable  $s = \cos \theta$ . The polar angle part of the Klein Gordon equation in (26) can be written in the following universal associated-Legendre differential equation form [38-40]

$$\frac{d^2 H(s)}{ds^2} - \frac{2s}{1-s^2} \frac{dH(s)}{ds} + \frac{1}{(1-s^2)^2} [j(j+D-2)(1-s^2) - m^2 - C\alpha_2^2 s^2] H(\theta) = 0. \quad (27)$$

Equation (27) has already been solved for the three-dimensional Schrödinger equation through the NU method in [38]. However, the aim in this subsection is to solve with different parameters resulting from the  $D$ -space-dimensions of Klein-Gordon equation. Further, Eq. (27) is compared with (14) and the following identifications are obtained

$$\tilde{\tau}(s) = -2s, \quad \sigma(s) = 1 - s^2, \quad \tilde{\sigma}(s) = -m'^2 + (1 - s^2)\nu', \quad (28)$$

where

$$\nu' = j'(j' + D - 2) = j(j + D - 2) + C\alpha_2^2, \quad m'^2 = m^2 + C\alpha_2^2. \quad (29)$$

Inserting the above expressions into equation (22), one obtains the following function:

$$\pi(s) = \pm \sqrt{(\nu' - k)s^2 + k - \nu' + m'^2}, \quad (30)$$

Following the method, the polynomial  $\pi(s)$  is found in the following possible values

$$\pi(s) = \begin{cases} m's & \text{for } k_1 = \nu' - m'^2, \\ -m's & \text{for } k_1 = \nu' - m'^2, \\ m' & \text{for } k_2 = \nu', \\ -m' & \text{for } k_2 = \nu'. \end{cases} \quad (31)$$

Imposing the condition  $\tau'(s) < 0$ , for equation (18), one selects

$$k_1 = \nu' - m'^2 \quad \text{and} \quad \pi(s) = -m's, \quad (32)$$

which yields

$$\tau(s) = -2(1 + m')s. \quad (33)$$

Using equations (19) and (23), the following expressions for  $\lambda$  are obtained, respectively,

$$\lambda = \lambda_n = 2\tilde{n}(1 + m') + \tilde{n}(\tilde{n} - 1), \quad (34)$$

$$\lambda = \nu' - m'(1 + m'). \quad (35)$$

We compare equations (34) and (35), the new angular momentum  $j$  values are obtained as

$$j = -\frac{(D-2)}{2} + \frac{1}{2}\sqrt{(D-2)^2 + (2\tilde{n} + 2m' + 1)^2 - 4C\alpha_2^2 - 1}, \quad (36)$$

or

$$j' = -\frac{(D-2)}{2} + \frac{1}{2}\sqrt{(D-2)^2 + (2\tilde{n} + 2m' + 1)^2 - 1}. \quad (37)$$

Using Eqs (15)-(17) and (20)-(21), the polynomial solution of  $y_n$  is expressed in terms of Jacobi polynomials [39,40] which are one of the orthogonal polynomials:

$$H_{\tilde{n}}(\theta) = N_{\tilde{n}} \sin^{m'}(\theta) P_{\tilde{n}}^{(m', m')}(\cos \theta), \quad (38)$$

where  $N_{\tilde{n}} = \frac{1}{2^{m'}(\tilde{n}+m')!} \sqrt{\frac{(2\tilde{n}+2m'+1)(\tilde{n}+2m')!n!}{2}}$  is the normalization constant determined by  $\int_{-1}^{+1} [H_{\tilde{n}}(s)]^2 ds = 1$  and using the orthogonality relation of Jacobi polynomials [35,45,46].

Besides

$$\tilde{n} = -\frac{(1 + 2m')}{2} + \frac{1}{2}\sqrt{(2j + 1)^2 + 4j(D - 3) + 4C\alpha_2^2}, \quad (39)$$

where  $m'$  is defined by equation (29).

### C. Eigensolutions of the radial equation

The solution of the radial part of Klein-Gordon equation, Eq. (24), for the Kratzer's potential has already been solved by means of NU-method in [39]. Very recently, using the same method, the problem for the non-central potential in (2) has been solved in three

dimensions ( $3D$ ) by Cheng and Dai [36]. However, the aim of this subsection is to solve the problem with a different radial separation function  $g(r)$  in any arbitrary dimensions. In what follows, we present the exact bound-states (real) solution of Eq. (24). Letting

$$\varepsilon^2 = \alpha_1^2 \alpha_2^2, \quad 4\gamma^2 = (M-1)(M-3) + 4B\alpha_2^2, \quad \beta^2 = A\alpha_2^2, \quad (40)$$

and substituting these expressions in equation (24), one gets

$$\frac{d^2 g(r)}{dr^2} + \left( \frac{-\varepsilon^2 r^2 + \beta^2 r - \gamma^2}{r^2} \right) g(r) = 0. \quad (41)$$

To apply the conventional NU-method, equation (41) is compared with (14), resulting in the following expressions:

$$\tilde{\tau}(r) = 0, \quad \sigma(r) = r, \quad \tilde{\sigma}(r) = -\varepsilon^2 r^2 + \beta^2 r - \gamma^2. \quad (42)$$

Substituting the above expressions into equation (22) gives

$$\pi(r) = \frac{1}{2} \pm \frac{1}{2} \sqrt{4\varepsilon^2 r^2 + 4(k - \beta^2)r + 4\gamma^2 + 1}. \quad (43)$$

Therefore, we can determine the constant  $k$  by using the condition that the discriminant of the square root is zero, that is

$$k = \beta^2 \pm \varepsilon \sqrt{4\gamma^2 + 1}, \quad 4\gamma^2 + 1 = (D + 2j - 2)^2 + 4B\alpha_2^2. \quad (44)$$

In view of that, we arrive at the following four possible functions of  $\pi(r)$  :

$$\pi(r) = \begin{cases} \frac{1}{2} + \left[ \varepsilon r + \frac{1}{2} \sqrt{4\gamma^2 + 1} \right] & \text{for } k_1 = \beta^2 + \varepsilon \sqrt{4\gamma^2 + 1}, \\ \frac{1}{2} - \left[ \varepsilon r + \frac{1}{2} \sqrt{4\gamma^2 + 1} \right] & \text{for } k_1 = \beta^2 + \varepsilon \sqrt{4\gamma^2 + 1}, \\ \frac{1}{2} + \left[ \varepsilon r - \frac{1}{2} \sqrt{4\gamma^2 + 1} \right] & \text{for } k_2 = \beta^2 - \varepsilon \sqrt{4\gamma^2 + 1}, \\ \frac{1}{2} - \left[ \varepsilon r - \frac{1}{2} \sqrt{4\gamma^2 + 1} \right] & \text{for } k_2 = \beta^2 - \varepsilon \sqrt{4\gamma^2 + 1}. \end{cases} \quad (45)$$

The correct value of  $\pi(r)$  is chosen such that the function  $\tau(r)$  given by Eq. (18) will have negative derivative [29]. So we can select the physical values to be

$$k = \beta^2 - \varepsilon \sqrt{4\gamma^2 + 1} \quad \text{and} \quad \pi(r) = \frac{1}{2} - \left[ \varepsilon r - \frac{1}{2} \sqrt{4\gamma^2 + 1} \right], \quad (46)$$

which yield

$$\tau(r) = -2\varepsilon r + (1 + \sqrt{4\gamma^2 + 1}), \quad \tau'(r) = -2\varepsilon < 0. \quad (47)$$

Using Eqs (19) and (23), the following expressions for  $\lambda$  are obtained, respectively,

$$\lambda = \lambda_n = 2n\varepsilon, \quad n = 0, 1, 2, \dots, \quad (48)$$

$$\lambda = \delta^2 - \varepsilon(1 + \sqrt{4\gamma^2 + 1}). \quad (49)$$

So we can obtain the Klein Gordon energy eigenvalues from the following relation:

$$\left[ 1 + 2n + \sqrt{(D + 2j - 2)^2 + 4(\mu + E_R)B} \right] \sqrt{\mu - E_R} = A\sqrt{\mu + E_R}, \quad (50)$$

and hence for the Kratzer plus the new ring-shaped potential, it becomes

$$\left[ 1 + 2n + \sqrt{(D + 2j - 2)^2 + 4a_0r_0^2(\mu + E_R)} \right] \sqrt{\mu - E_R} = 2a_0r_0\sqrt{\mu + E_R}, \quad (51)$$

with  $j$  defined in (36). Although Eq. (51) is exactly solvable for  $E_R$  but it looks to be little complicated. Further, it is interesting to investigate the solution for the Coulomb potential. Therefore, applying the following transformations:  $A = Ze^2$ ,  $B = 0$ , and  $j = \ell$ , the central part of the potential in (3) turns into the Coulomb potential with Klein Gordon solution for the energy spectra given by

$$E_R = \mu \left( 1 - \frac{2q^2e^2}{q^2e^2 + (2n + 2\ell + D - 1)^2} \right), \quad n, \ell = 0, 1, 2, \dots, \quad (52)$$

where  $q = Ze$  is the charge of the nucleus. Further, Eq. (52) can be expanded as a series in the nucleus charge as

$$E_R = \mu - \frac{2\mu q^2e^2}{(2n + 2\ell + D - 1)^2} + \frac{2\mu q^4e^4}{(2n + 2\ell + D - 1)^4} - O(qe)^6, \quad (53)$$

The physical meaning of each term in the last equation was given in detail by Ref. [36]. Besides, the difference from the conventional nonrelativistic form is because of the choice of the vector  $V(r, \theta, \varphi)$  and scalar  $S(r, \theta, \varphi)$  parts of the potential in Eq. (1).

Overmore, if the value of  $j$  obtained by Eq.(36) is inserted into the eigenvalues of the radial part of the Klein Gordon equation with the noncentral potential given by Eq. (51), we finally find the energy eigenvalues for a bound electron in the presence of a noncentral potential by Eq. (2) as

$$\left[1 + 2n + \sqrt{(2j' + D - 2)^2 + 4(a_0 r_0^2 - C)(\mu + E_R)}\right] \sqrt{\mu - E_R} = 2a_0 r_0 \sqrt{\mu + E_R}, \quad (54)$$

where  $m' = \sqrt{m^2 + C(\mu + E_R)}$  and  $\tilde{n}$  is given by Eq. (39). On the other hand, the solution of the Schrödinger equation, Eq. (9), for this potential has already been obtained by using the same method in Ref. [39] and it is in the Coulombic-like form:

$$E_{NR} = -\frac{8\mu a_0^2 r_0^2}{\left[2n + 1 + \sqrt{(2\ell' + D - 2)^2 + 8\mu(a_0 r_0^2 - C)}\right]^2}, \quad n = 0, 1, 2, \dots \quad (55)$$

$$2\ell' + D - 2 = \sqrt{(D - 2)^2 + (2\tilde{n} + 2m' + 1)^2} - 1, \quad (56)$$

where  $m' = \sqrt{m^2 + 2\mu C}$ . Also, applying the following appropriate transformation:  $\mu + E_R \rightarrow 2\mu$ ,  $\mu - E_R \rightarrow -E_{NR}$ ,  $j \rightarrow \ell$  to Eq. (54) provides exactly the nonrelativistic limit given by Eq. (55).

In what follows, let us now turn attention to find the radial wavefunctions for this potential. Substituting the values of  $\sigma(r)$ ,  $\pi(r)$  and  $\tau(r)$  in Eqs (42), (45) and (47) into Eqs. (17) and (21), we find

$$\phi(r) = r^{(\zeta+1)/2} e^{-\varepsilon r}, \quad (57)$$

$$\rho(r) = r^\zeta e^{-2\varepsilon r}, \quad (58)$$

where  $\zeta = \sqrt{4\gamma^2 + 1}$ . Then from equation (20), we obtain

$$y_{nj}(r) = B_{nj} r^{-\zeta} e^{2\varepsilon r} \frac{d^n}{dr^n} \left( r^{n+\zeta} e^{-2\varepsilon r} \right), \quad (59)$$

and the wave function  $g(r)$  can be written in the form of the generalized Laguerre polynomials as

$$g(\rho) = C_{nj} \left( \frac{\rho}{2\varepsilon} \right)^{(1+\zeta)/2} e^{-\rho/2} L_n^\zeta(\rho), \quad (60)$$

where for Kratzer's potential we have

$$\zeta = \sqrt{(D+2j-2)^2 + 4a_0r_0^2(\mu + E_R)}, \quad \rho = 2\varepsilon r. \quad (61)$$

Finally, the radial wave functions of the Klein-Gordon equation are obtained

$$R(\rho) = C_{nj} \left( \frac{\rho}{2\varepsilon} \right)^{(\zeta+2-D)/2} e^{-\rho/2} L_n^\zeta(\rho), \quad (62)$$

where  $C_{nj}$  is the normalization constant to be determined below. Using the normalization condition,  $\int_0^\infty R^2(r)r^{D-1}dr = 1$ , and the orthogonality relation of the generalized Laguerre polynomials,  $\int_0^\infty z^{\eta+1}e^{-z} [L_n^\eta(z)]^2 dz = \frac{(2n+\eta+1)(n+\eta)!}{n!}$ , we have

$$C_{nj} = \left( 2\sqrt{\mu^2 - E_R^2} \right)^{1+\frac{\zeta}{2}} \sqrt{\frac{n!}{(2n+\zeta+1)(n+\zeta)!}}. \quad (63)$$

Finally, we may express the normalized total wave functions as

$$\begin{aligned} \psi(r, \theta, \varphi) = & \frac{\left( 2\sqrt{\mu^2 - E_R^2} \right)^{1+\frac{\zeta}{2}}}{2^{m'}(\tilde{n}+m')!} \sqrt{\frac{(2\tilde{n}+2m'+1)(\tilde{n}+2m')!\tilde{n}!n!}{2\pi(2n+\zeta+1)(n+\zeta)!}} \\ & \times r^{\frac{(\zeta+2-D)}{2}} \exp(-\sqrt{\mu^2 - E_R^2}r) L_n^\zeta(2\sqrt{\mu^2 - E_R^2}r) \sin^{m'}(\theta) P_n^{(m',m')}(\cos \theta) \exp(\pm im\varphi). \end{aligned} \quad (64)$$

where  $\zeta$  is defined in Eq. (61) and  $m'$  is given after the Eq. (54).

## V. CONCLUSIONS

The relativistic spin-0 particle  $D$ -dimensional Klein-Gordon equation has been solved easily for its exact bound-states with equal scalar and vector ring-shaped Kratzer potential through the conventional NU method. The analytical expressions for the total energy levels and eigenfunctions of this system can be reduced to their conventional three-dimensional space form upon setting  $D = 3$ . Further, the noncentral potentials treated in [30] can be introduced as perturbation to the Kratzer's potential by adjusting the strength of the

coupling constant  $C$  in terms of  $a_0$ , which is the coupling constant of the Kratzer's potential. Additionally, the radial and polar angle wave functions of Klein-Gordon equation are found in terms of Laguerre and Jacobi polynomials, respectively. The method presented in this paper is general and worth extending to the solution of other interaction problems. This method is very simple and useful in solving other complicated systems analytically without given a restriction conditions on the solution of some quantum systems as the case in the other models. We have seen that for the nonrelativistic model, the exact energy spectra can be obtained either by solving the Schrödinger equation in (9) (cf. Ref. [39] or Eq. (55)) or by applying appropriate transformation to the relativistic solution given by Eq. (54). Finally, we point out that these exact results obtained for this new proposed form of the potential (2) may have some interesting applications in the study of different quantum mechanical systems, atomic and molecular physics.

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